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2002 J. Phys. A: Math. Gen. 35 L141

(http://iopscience.iop.org/0305-4470/35/11/102)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) L141–L146 PII: S0305-4470(02)31994-2

LETTER TO THE EDITOR

Is there a true model-D critical dynamics?

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Received 17 December 2001
Published 8 March 2002
Online at stacks.iop.org/JPhysA/35/L141

Abstract
We show that non-locality in the conservation of both the order parameter and a non-critical density (model-D dynamics) leads to new fixed points for critical dynamics. Depending upon the parameters characterizing the non-locality in the two fields, we find four regions: (i) model-A-like, where both conservations are irrelevant; (ii) model-B-like, with the conservation in the order parameter field relevant and the conservation in the coupling field irrelevant; (iii) model-C-like, where the conservation in the order parameter field is irrelevant but the conservation in the coupling field is relevant; and (iv) model-D-like, where both conservations are relevant. While the first three behaviours are already known in dynamical critical phenomena, the last one is a novel phenomenon due entirely to the non-locality in the two fields.

PACS numbers: 64.60.Ht, 05.10.Cc, 45.30.+s

On the basis of long-time behaviour, the dynamics of different critical systems are classified [1] into various dynamic universality classes. These classes are characterized by several factors, such as the number of conserved fields and the non-vanishing Poisson brackets, in addition to the usual quantities necessary to characterize equilibrium behaviour. Among the models which have no propagating modes, model A is a system in which there are no conserved fields, model B is a system with a conserved order parameter, while model C is the class of systems which have a non-conserved order parameter field coupled to a conservative field. A fourth class was also identified—the model-D system, in which both the order parameter field and the coupled field are conserved. Model D, however, was shown to be equivalent to model B, as the coupling to the field becomes irrelevant in the large-length long-timescale limit.

The dynamical behaviour in systems with one or more conserved fields, like the systems of models B, C, and D in the above classification, is usually studied by considering local conservation. However, conservation is by no means necessarily local and, in fact, non-local conservation is a more general condition. A few recent studies on critical or near-critical dynamics show that non-locality in the conserved field may change the dynamical class, besides leading to the obvious modifications in non-universal details. This kind of effect of non-locality in the conservation has been noted in the study of critical dynamics of models B and C and in the numerical study of the early-time effect of a model believed to belong to the class of model C, and on the zero-temperature dynamics (phase-ordering kinetics or Ostwald ripening) [2–7]. Besides the theoretical interest in its own right, one of the important applications of non-local dynamics is in speeding up numerical simulations, prompting implementation of various types of non-local move [8]. Hence the need for a general classification of effects of non-locality in dynamics.

In a sense, the model-D class is the most generalized class of dynamical critical phenomena with dissipative dynamics. However, with local conservation, it is equivalent to the model-B class showing no novel features. Motivated by the results for model C [4], we investigate the problem of non-local conservation in model D in the most general way by keeping both conservations—that of the order parameter field and that of the coupled field—non-local. One of our main results is that a true model-D behaviour emerges in the non-local dynamics, unlike in the local conservation case.

We introduce non-locality by using a non-local kernel [2–4] in the chemical potential, keeping the continuity equation intact. Using ϕ for the order parameter and m for the secondary coupling parameter, the Hamiltonian of model D is given by

$$H = \int \mathrm{d}^d \boldsymbol{x}\, [\tfrac{1}{2} r\phi^2(\boldsymbol{x}) + \tfrac{1}{2}(\nabla\phi(\boldsymbol{x}))^2 + \tilde{u}\phi^4(\boldsymbol{x}) + \gamma\phi^2(\boldsymbol{x})m(\boldsymbol{x}) + \tfrac{1}{2}C^{-1}m^2(\boldsymbol{x})], \tag{1}$$

where $\phi^2(\boldsymbol{x}) = \sum_{i=1}^n \phi_i^2(\boldsymbol{x})$, $\phi^4(\boldsymbol{x}) = [\sum_{i=1}^n \phi_i^2(\boldsymbol{x})]^2$, $r = 0$ is the mean field critical point, and $C > 0$ ensures non-criticality of m. There is a short-distance cut-off which in momentum space is an ultraviolet cut-off Λ. We shall set $\Lambda = 1$ in the calculation. The above Hamiltonian, equation (1), is the same as in model C [1, 4]. Making $\gamma = 0$ effectively reduces the above model to model A.

The dynamics[3] is given by the following equations:

$$\left(\frac{1}{\Gamma_\rho k^\rho} + \frac{1}{\Gamma_0}\right)\frac{\partial\phi_k}{\partial t} = -\frac{\delta H}{\delta\phi_{-k}} + \eta(\boldsymbol{k}, t) \tag{2a}$$

$$\left(\frac{1}{\lambda_\sigma k^\sigma} + \frac{1}{\lambda_0}\right)\frac{\partial m_k}{\partial t} = -\frac{\delta H}{\delta m_{-k}} + \zeta(\boldsymbol{k}, t). \tag{2b}$$

The noises η and ζ obey

$$\langle\eta\rangle = 0, \qquad \langle\zeta\rangle = 0, \tag{3a}$$

$$\langle\eta(\boldsymbol{k}, t)\eta(\boldsymbol{k}', t')\rangle = 2\left(\frac{1}{\Gamma_\rho k^\rho} + \frac{1}{\Gamma_0}\right)\delta(\boldsymbol{k} + \boldsymbol{k}')\delta(t - t'), \tag{3b}$$

$$\langle\zeta(\boldsymbol{k}, t)\zeta(\boldsymbol{k}', t')\rangle = 2\left(\frac{1}{\lambda_\sigma k^\rho} + \frac{1}{\lambda_0}\right)\delta(\boldsymbol{k} + \boldsymbol{k}')\delta(t - t'). \tag{3c}$$

Here, $k = |\boldsymbol{k}|$, and we have introduced two different parameters ρ and σ to denote the non-localities in the order parameter field and the coupling field respectively. The dissipative and kinetic coefficients (Γ_0 and λ_0) are labelled by the subscript 0 while the transport coefficients (Γ_ρ and λ_σ) are labelled by the corresponding powers of k. For the local case, both ρ and σ assume a value equal to 2. Results are known for some limiting values of ρ and σ: (a) ($\rho = 0, \gamma = 0$) corresponds to model A [1]; (b) ($\rho = 2, \gamma = 0$) and ($\rho = 2, \sigma = 2$) give model-B-like behaviour [1]; (c) any value of $\rho > 0$ and $\gamma = 0$ give a non-local model B [2,5]; and (d) $\rho = 0$ with any $\sigma > 0$ gives non-local model C [4].

[3] Additional torques for $n = 3$ (Heisenberg model) may lead to new dynamic universality classes [9]. We do not consider such effects.

A Hamiltonian of the type of equation (1) occurs near the critical wings of a tricritical system or in critical systems with additional constraints. Model-D dynamics is relevant in such cases—for example, in three- or four-component mixtures or in spin-1 or Potts models (see e.g. [10]). One spin-$\frac{1}{2}$ example—although a bit unrealistic—is an antiferromagnetic system with conserved magnetization (as in model C [6, 11]) as well as conserved staggered magnetization. The Kawasaki dynamics can be used to conserve the magnetization, maintaining at the same time that the exchanges between opposite spins have to be done in such a way that the order parameter is unchanged.

A momentum-shell renormalization group approach is used to study the long-distance long-time behaviour. In this approach, small-scale fluctuations in space (between $\Lambda e^{-\delta l}$ and Λ in momentum space) are integrated out, and the cut-off rescaled to Λ. One then obtains an effective Hamiltonian and effective equations for the dynamics valid for longer scales. The universal features are obtained from the fixed points of the flow equations for the various parameters of the problem. As in the usual RG scheme, any parameter that vanishes (grows) in the long-scale limit is called an irrelevant (relevant) variable. A fixed point is then characterized by the set of relevant parameters. In this scheme the four dynamics classes correspond to the relevance and irrelevance of λ_σ and Γ_ρ as mentioned in the abstract.

The statics of this system is the same as for model C and the fixed-point values of the parameters r, u, and $\gamma^2 C$ are as given in [4]. The dynamical equations, on the other hand, take the forms

$$\frac{\partial \lambda_\sigma^{-1}}{\partial l} = \left(-z + \sigma + \frac{\alpha}{\nu}\right)\lambda_\sigma^{-1} \tag{4a}$$

$$\frac{\partial \lambda_0^{-1}}{\partial l} = \left(-z + \frac{\alpha}{\nu}\right)\lambda_0^{-1} + (n\gamma^2 K_d)Q \tag{4b}$$

$$\frac{\partial \Gamma_\rho^{-1}}{\partial l} = (-z + \rho + 2)\Gamma_\rho^{-1} \tag{4c}$$

$$\frac{\partial \Gamma_0^{-1}}{\partial l} = (-z + 2)\Gamma_0^{-1} + (4C\gamma^2 K_d)Q\frac{1}{1 + (Q/CP)} \tag{4d}$$

where $Q = \Gamma_\rho^{-1} + \Gamma_0^{-1}$, $P = \lambda_\sigma^{-1} + \lambda_0^{-1}$, and $K_d = 2^{1-d}\pi^{-d/2}\Gamma(d/2)$. The parameters λ_σ, λ_0, Γ_ρ, Γ_0, γ, and C appearing in the above equations are all l-dependent. To simplify notation, this l-dependence has been suppressed.

From (4b) we get

$$\frac{\Gamma_0}{\lambda_0} = \frac{(1+x)n\gamma^2 K_d}{z - \alpha/\nu} \tag{5}$$

where $x = \Gamma_0/\Gamma_\rho$ is the dimensionless ratio of the two coefficients for the primary order parameter (note: $\Lambda = 1$). This quantity x satisfies

$$\frac{\partial x}{\partial l} = (\rho - Y)x, \qquad \text{where } Y = \frac{2\alpha}{\nu n}\,\frac{1}{\mu^{-1} + (x+1)^{-1}}. \tag{6}$$

Here we have introduced the ratios of the transport and kinetic coefficients of m with the kinetic coefficient of ϕ:

$$\mu_\sigma = \frac{\Gamma_0 C}{\lambda_\sigma}, \qquad \mu_0 = \frac{\Gamma_0 C}{\lambda_0}, \qquad \text{and} \qquad \mu = \mu_\sigma + \mu_0.$$

By using the fixed-point value of C and equation (5), μ_0 can be rewritten as

$$\mu_0 = \frac{(x+1)\alpha/\nu}{2(z - \alpha/\nu)}.$$

The flow equation for μ_σ is

$$\frac{\partial \mu_\sigma}{\partial l} = \mu_\sigma[\sigma - 2 + \alpha/\nu - Y]. \tag{7}$$

We consider the different solutions of the equations for x and μ_σ. Corresponding to the relevance of the conserved quantities, one gets different regions in the ρ–σ plane with different dynamic critical behaviours. It should be mentioned here that for $n \geqslant 4$, the coupling γ scales to zero and one effectively gets a non-local model B. This has already been considered and shown to be model-A-like when globally conserved [2]. Our discussions are for the $n < 4$ region where the coupling survives.

There are three fixed points of x: $x = 0$, $x = \infty$, and a non-zero finite fixed-point value for x from equation (6):

Case 1. $x = 0$ in the long-length-scale limit. This corresponds to non-local model C: the conservation of the order parameter is irrelevant. The fixed points and their stabilities of non-local model C, which were obtained in [4] from the solutions for μ_σ, now depend on the value of ρ. In general, $x = 0$ will be a stable fixed point as long as $\rho < Y$. We focus on the stability of the various regions for non-zero values of ρ.

(a) μ_σ has a finite fixed-point value and $z = \sigma + \alpha/\nu$. This is a stable solution in the region $(2/n) - 1 > p > -1$ (where $p = (\sigma - 2)/(\alpha/\nu)$) for $\rho = 0$. For non-zero ρ this remains stable for $\rho < p + 1$.

(b) $\mu_\sigma = \infty$: this is valid for $p > (2/n) - 1$. Here $z = 2 + 2\alpha/n\nu$. This solution $x = 0$, $\mu_\sigma = \infty$ is stable as long as $\rho < 2\alpha/n\nu$.

(c) For $\mu_\sigma = 0$, the stability of $x = 0$ is valid only for $p < -1$ and $\rho < \mathrm{O}(\epsilon^2)$. Here $z = 2 + \mathrm{O}(\epsilon^2)$ and the behaviour is model-A-like. Any non-zero value of ρ thus destroys this model-A-like region. Therefore, a region where both the conservations are irrelevant is restricted strictly to $\rho = 0$.

Case 2. A finite non-zero value of x in the long-length-scale limit. This occurs when $\rho = Y$. In order to ensure that Γ_0 reaches a finite fixed-point value, we must have $z = 2 + \rho$. In this case, μ_σ again has the following fixed points:

(a) $\mu_\sigma = \infty$ for $\sigma - 2 + \alpha/\nu > \rho$;

(b) $\mu_\sigma = 0$ for $\sigma - 2 + \alpha/\nu < \rho$;

(c) μ_σ non-zero and finite for $\sigma - 2 + \alpha/\nu = \rho$.

Physically, a finite-valued fixed point of x means that the conservation in the order parameter ϕ is relevant. If at the same time μ_σ attains a non-zero fixed-point value, the conservation of the coupling field m is also relevant. This occurs if the fixed points (a) and (c) are stable.

For case (a), $x = ((\rho - 2\alpha)/\nu n)/(2\alpha/\nu n)$, and for non-negative values of x, $\rho > 2\alpha/n\nu$. Hence between $\rho = 2\alpha/n\nu$ and $\rho = \sigma - 2 + \alpha/\nu$, a stable region is obtained where both conservations are relevant.

For case (b), $x + 1 = \rho(2z - \alpha/\nu)/(2\alpha^2/n\nu^2)$. This has a finite-valued solution for x only for $\rho \sim \mathrm{O}(\epsilon^2)$. Hence for $\rho > \sigma - 2 + \alpha/\nu$, $x \to \infty$ is the only solution for $\mu = 0$. It is a conventional model-B fixed point.

For case (c), $z = \sigma + \alpha/\nu$ and, in principle, along the entire line $\sigma - 2 + \alpha/\nu = \rho$ a solution exists where both the conservations are relevant. However, there is a restriction on the value of σ arising from the condition of existence of a non-zero finite value of μ_σ. We find that

$$\mu_\sigma + \mu_0 = \frac{(\sigma - 2 + \alpha/\nu)(x + 1)}{2 - \sigma - \alpha/\nu + (2\alpha/n\nu)(x + 1)}. \tag{8}$$

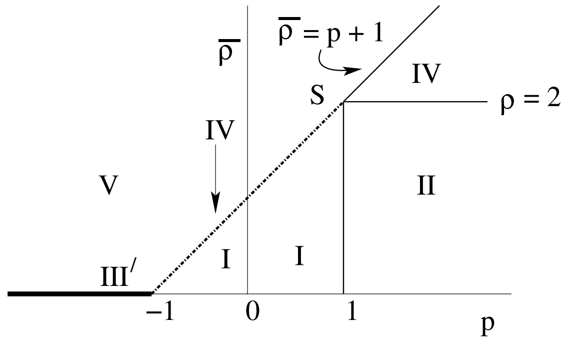

Figure 1. The different regions in the non-local model D with $n = 1$ are shown in the rescaled $\bar{\rho}$–p plane where $\bar{\rho} = \rho/(\alpha/\nu)$ and $p = (\sigma - 2)/(\alpha/\nu)$. Regions I and II are model-C-like with $z = \sigma + \alpha/\nu$ and $z = 2 + 2\alpha/\nu$ respectively. Region III′ is model-A-like (as in model C of [4]) with $z = 2 + O(\epsilon^2)$. Region IV is the model-D region, with $z = \rho + 2$. Region V is model-B-like, with $z = \rho + 2$. The line $p + 1 = \rho$ is also model-D-like up to point S (shown by the dashed line). There are no discontinuities of z along any boundary except at the boundary between III′ and V.

Hence with $\sigma = 2 + p\alpha/\nu$, we get the condition $(1 + p) < 2(x + 1)/n$. Since x is also non-zero, this implies that this solution with both x and μ_σ finite is valid for $p + 1 = \rho/(\alpha/\nu)$ with at least $p < (2/n) - 1$.

In the rest of the ρ–σ plane, the only solution is $x = \infty$ and $1/\lambda_\sigma = 0$, for which only the conservation of the order parameter is relevant. Here $z = \rho + 2$.

All the above possibilities for different dynamic behaviours are summarized in figure 1 in the ρ–σ plane (for $n = 1$). The possibility of seeing a region where both conservation conditions are relevant raises new issues—such as the early-time effect and boundary or surface effects (see e.g. [6, 12])—in the new regime. These remain to be studied.

To summarize, model D can be viewed as the most general dynamic model with purely dissipative dynamics and we do indeed obtain all four types of behaviour as soon as non-locality in the conservations (as in equation (2)) is introduced. Since our general model subsumes the previously studied non-local model C and local model D, it is natural to expect A-like, B-like, and C-like regimes. The model-A-like region, however, turns out to be unstable for any finite value of ρ. The model-C-like regions are stable for small non-zero values of ρ but ultimately disappear for larger values of ρ. The most non-trivial result, for general values of ρ and σ, is the appearance (region IV in figure 1) of a region with a new dynamical behaviour where both conservations are relevant. This we call a *true model-D-like region*—a consequence of non-local conservation laws.

PS acknowledges financial support from DST project SP/S2/M-11/99.

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